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Analysing the structure of the integrating factors for first-order ordinary differential equations with Liouvillian functions in the solution

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Abstract
Here we demonstrate a theorem concerning the general structure of the integrating factor for first-order ordinary differential equations whose solutions contain Liouvillian functions. This result assures the generality of a method presented in a forthcoming paper extending the usual Prelle–Singer approach.

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1. Introduction

When talking about solving a differential equation many ideas come to mind. For a first-order ordinary differential equation (FOODE), finding the solution can be equated to determining an integrating factor.

A remarkable method for finding such factors was developed, in 1983, by Prelle and Singer [1]. Their method is based on the knowledge of the general structure of the integrating factor for FOODEs of the type $\mathrm{d}y/\mathrm{d}x = M(x, y)/N(x, y)$, with M and N polynomials in their arguments, which present a solution that can be written in terms of elementary functions[3]. Their approach is very attractive due to the fact that it is non-classificatory and of a semi-decision nature. Therefore, it has motivated many extensions of the original idea [3–6].

In this paper, we take a further step in establishing the general structure of the integrating factor for FOODEs of type $\mathrm{d}y/\mathrm{d}x = M(x, y)/N(x, y)$, with M and N polynomials in their arguments, which present a solution that can be written in terms of Liouvillian functions[4] (LFOODEs). This result can be used to assure the applicability of the method

[3] For a formal definition of elementary function, see [2].
[4] An extension of elementary functions, see [2].

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presented in [7], which is an extension to the Prelle–Singer (PS) procedure allowing for the solution of some LFOODEs missed by the usual PS procedure.

The paper is organized as follows: in section 2.1, we summarize some earlier results concerning the structure of the integrating factors for some classes of LFOODEs; next, in section 2.2, we present a theorem confirming the above-mentioned conjecture and then present our conclusions.

2. The structure of the integrating factor for LFOODEs

2.1. First results

A seminal result on dealing with LFOODEs was obtained by Prelle and Singer in 1983 [1]. They have demonstrated that, for an LFOODE,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{M(x, y)}{N(x, y)} \tag{1}$$

where M and N are polynomials in (x, y) with coefficients in the complex field C, if its solution can be written in terms of elementary functions, then there exists an integrating factor of the form $R = \prod_i f_i^{n_i}$ where f_i are irreducible polynomials and n_i are non-zero rational numbers. Using this result in (1), we have

$$\frac{D[R]}{R} = \sum_i \frac{n_i D[f_i]}{f_i} = -(\partial_x N + \partial_y M) \tag{2}$$

where $D \equiv N\partial_x + M\partial_y$.

From (2), plus the fact that M and N are polynomials, we conclude that $D[R]/R$ is a polynomial and that $f_i|D[f_i]$ [1]. We now have a criterion for choosing the possible f_i (build all the possible divisors of $D[f_i]$ up to a certain degree) and, if we manage to solve (2), thereby finding n_i, we know the integrating factor for the FOODE and the problem is reduced to a quadrature.

In [7, 8], a next step was taken: it was shown that, for an LFOODE of type (1), the integrating factor is of the form

$$R = \mathrm{e}^{r_0(x,y)} \prod_{i=1}^{n} p_i(x, y)^{c_i} \tag{3}$$

where r_0 is a rational function of (x, y), the p_i are irreducible polynomials in (x, y) and the c_i are constants.

So, it is straightforward to see that an LFOODE of the type (1), which presents an integrating factor with $r_0 \neq$ constant, is beyond the scope of the PS-method.

2.2. A theorem

Theorem 1. *If we have an LFOODE of the form* $\mathrm{d}y/\mathrm{d}x = M(x, y)/N(x, y)$*, where M and N are polynomials in (x, y), with integrating factor R given by* $R = \mathrm{e}^{r_0(x,y)} \prod_{i=1}^{n} p_i(x, y)^{c_i}$*, where r_0 is a rational function of (x, y), p_i are irreducible polynomials in (x, y) and c_i are constants, then* $D[r_0]$ *is a polynomial in (x, y), where $D \equiv N\partial_x + M\partial_y$.*

Proof. Applying (3) to equation (2), we get

$$D[r_0] + \sum_i \frac{c_i D[p_i]}{p_i} = -(\partial_x N + \partial_y M). \tag{4}$$

Since r_0 is a rational function, we can write (4) as

$$D\left[\frac{P(x,y)}{Q(x,y)}\right] + \sum_i c_i \frac{D[p_i]}{p_i} = -(\partial_x N + \partial_y M) \tag{5}$$

where P and Q are polynomials in (x, y) with no common factors. Writing $\sum_i c_i \frac{D[p_i]}{p_i}$ as a single quotient, we get

$$D\left[\frac{P}{Q}\right] + \frac{\sum_j c_j (\prod_{i,i\neq j} p_i) D[p_j]}{\prod_i p_i} = -\left(\partial_x N + \partial_y M\right). \tag{6}$$

Expanding $D\left[\frac{P}{Q}\right]$ and multiplying both sides of (6) by $\prod_i p_i$, we can write

$$\prod_i p_i \; \frac{Q\,D[P] - P\,D[Q]}{Q^2} + \sum_j c_j \left(\prod_{i,i\neq j} p_i\right) D[p_j] = -\left(\partial_x N + \partial_y M\right)\left(\prod_i p_i\right). \tag{7}$$

Since D is a linear differential operator, with polynomial coefficients, and the p_i are polynomials, the $D[p_i]$ are also polynomial. Therefore, $\sum_j c_j (\prod_{i,i\neq j} p_i) D[p_j]$ is polynomial and so is the right-hand side of (7). From this, we can conclude that the term

$$\prod_i p_i \; \frac{Q\,D[P] - P\,D[Q]}{Q^2} \tag{8}$$

is polynomial.

Noting that $Q\,D[P] - P\,D[Q]$ is polynomial and the p_i are independent irreducible polynomials, $\prod_i p_i$ cannot cancel Q^2 out (i.e. $\prod_i p_i / Q^2$ cannot be polynomial). So, we have two possible situations:

- $\prod_i p_i$ and Q have no common factors;
- $\prod_i p_i$ and Q have common factors.

(1) First situation. Since $\prod_i p_i$ does not have any common factor with Q (so has no common factor with Q^2 either) and $\prod_i p_i \; \frac{Q\,D[P]-P\,D[Q]}{Q^2}$ is polynomial, we must have that

$$D[r_0] = \frac{Q\,D[P] - P\,D[Q]}{Q^2} \tag{9}$$

is itself a polynomial, as we wanted to demonstrate.

(2) Second situation. This case is a little more involved. First, let us consider that, in $\prod_i p_i$, i runs from 1 to n. With that in mind, let us establish some notation.

Representing the common factor of Q and $\prod_{i=1}^n p_i$ as

$$\mathcal{I} = \prod_{i=1}^{n_\mathcal{I}} p_i \tag{10}$$

and the terms in $\prod_{i=1}^n p_i$ not present in Q as

$$\pi = \prod_{i=n_\mathcal{I}+1}^{n} p_i \tag{11}$$

we can write

$$\prod_{i=1}^{n} p_i = \pi\,\mathcal{I}. \tag{12}$$

Recalling that Q is polynomial, it can be written as a product of powers of irreducible polynomials. Since, by assumption, Q has a common factor $\mathcal{I}$ with $\prod_{i=1}^{n} p_i$, we are going to write

$$Q = \theta\ \mathcal{I} = \left(\prod_{i=1}^{n_\theta} q_i^{m_i}\right)\left(\prod_{i=1}^{n_\mathcal{I}} p_i\right) \tag{13}$$

where q_i are irreducible polynomials and m_i are positive integers[5].

Re-writing (8) with this notation and expanding, we obtain

$$\begin{aligned}\prod_i p_i\ \frac{Q\,D[P] - P\,D[Q]}{Q^2} &= (\pi\ \mathcal{I})\ \frac{Q\,D[P] - P\,D[Q]}{Q\,\theta\,\mathcal{I}}\\ &= \pi\ \frac{Q\,D[P] - P\,D[Q]}{Q\,\theta} = \pi\ \frac{D[P]}{\theta} - \pi\ P\,\frac{D[Q]}{Q\,\theta}.\end{aligned} \tag{14}$$

Remembering that the term (14) is a polynomial, if we multiply it by θ (itself a polynomial, see (13)), we get that

$$\pi\ D[P] - \pi\ P\,\frac{D[Q]}{Q} \tag{15}$$

is a polynomial. Therefore, since $\pi\ D[P]$ is a polynomial, we finally may conclude that

$$\pi\ P\,\frac{D[Q]}{Q} \tag{16}$$

is a polynomial. From the fact that neither π nor P have factors in common with Q, we can assure that $D[Q]/Q$ is a polynomial. Using this fact and denoting

$$Q = \prod_{i=1}^{n_q} \varrho_i^{k_i}\ (= \theta\ \mathcal{I}) \tag{17}$$

where the ϱ_i are irreducible polynomials and the k_i are integers, we have that

$$\frac{D[Q]}{Q} = \frac{D\left[\prod_{i=1}^{n_q} \varrho_i^{k_i}\right]}{\prod_{i=1}^{n_q} \varrho_i^{k_i}} = \sum_{i=1}^{n_q} k_i\ \frac{D[\varrho_i]}{\varrho_i}. \tag{18}$$

If we multiply (18) by $\prod_{j=2}^{n_q} \varrho_j$, we get

$$\left(\prod_{j=2}^{n_q} \varrho_j\right)\frac{D[Q]}{Q} = k_1\left(\prod_{j=2}^{n_q} \varrho_j\right)\frac{D[\varrho_1]}{\varrho_1} + \sum_{i=2}^{n_q} k_i\left(\prod_{j=2, j\neq i}^{n_q} \varrho_j\right) D[\varrho_i]. \tag{19}$$

Since the left-hand side of (19) and the second term on the right-hand side of (19) are polynomials, we may conclude that $k_1\left(\prod_{j=2}^{n_q} \varrho_j\right) D[\varrho_1]/\varrho_1$ is also a polynomial. Considering that the ϱ are independent (by construction), the product $\prod_{j=2}^{n_q} \varrho_j$ cannot cancel ϱ_1. Therefore, we can conclude that $\varrho_1 | D[\varrho_1]$. In an analogous way, we have that $\varrho_i | D[\varrho_i]$, $i = 2 \cdots n_q$. Finally, looking at (17) (noting that the ϱ are just another name for the q and p which build up Q), we can say that $q_i | D[q_i]$, $i = 1 \ldots n_\theta$ and $p_i | D[p_i]$, $i = 1 \ldots n_\mathcal{I}$.

Writing equation (4) as

$$D[r_0] + \sum_{i=1}^{n_\mathcal{I}} \frac{c_i\,D[p_i]}{p_i} + \sum_{i=n_\mathcal{I}+1}^{n} \frac{c_i\,D[p_i]}{p_i} = -(\partial_x N + \partial_y M) \tag{20}$$

[5] Note that some of the q_i may be the same irreducible polynomials that appear in $\mathcal{I}$.

and, since $p_i | D[p_i]$, $i = 1 \ldots n_{\mathcal{I}}$, we have that $\sum_{i=1}^{n_{\mathcal{I}}} \frac{c_i D[p_i]}{p_i}$ is a polynomial. Therefore

$$D[r_0] + \sum_{i=n_{\mathcal{I}}+1}^{n} \frac{c_i D[p_i]}{p_i} \tag{21}$$

is a polynomial. We can write (21) as

$$D\left[\frac{P}{Q}\right] + \frac{\sum_{j=n_{\mathcal{I}}+1}^{n} c_j \left(\prod_{i=n_{\mathcal{I}}+1, i\neq j}^{n} p_i\right) D[p_j]}{\prod_{i=n_{\mathcal{I}}+1}^{n} p_i}. \tag{22}$$

Multiplying (22) by $\prod_{i=n_{\mathcal{I}}+1}^{n} p_i$, we get

$$\prod_{i=n_{\mathcal{I}}+1}^{n} p_i \; \frac{Q\, D[P] - P\, D[Q]}{Q^2} + \sum_{j=n_{\mathcal{I}}+1}^{n} c_j \left(\prod_{i=n_{\mathcal{I}}+1, i\neq j}^{n} p_i \right) D[p_j] \tag{23}$$

which is also a polynomial. Since $\sum_j c_j (\prod_{i, i\neq j} p_i) D[p_j]$ is itself a polynomial, we can safely say that

$$\prod_{i=n_{\mathcal{I}}+1}^{n} p_i \; \frac{Q\, D[P] - P\, D[Q]}{Q^2} \tag{24}$$

is a polynomial. Since $\prod_{i=n_{\mathcal{I}}+1}^{n} p_i$ has no common factor with Q, we finally conclude that $\frac{Q\, D[P] - P\, D[Q]}{Q^2} = D[r_0]$ is polynomial, as we wanted to demonstrate. □

Corollary 1. *If* $R = \mathrm{e}^{r_0(x,y)} \prod_{i=1}^{n} p_i(x, y)^{c_i}$ *(where r_0 is a rational function of (x, y), the p_i are irreducible polynomials in (x, y) and the c_i are constants) is the integrating factor for the LFOODE* $\mathrm{d}y/\mathrm{d}x = M/N$*, where M, N are polynomials in (x, y), then $p_i | D[p_i]$.*

The result above is a direct consequence of equation (4) and the above theorem.

3. Conclusion

The result presented here is a step forward in the determination of the general form for the integrating factor for an LFOODE of the type given by equation (1). Now one can say that the integrating factor for such an LFOODE can be put in the form

$$R = \mathrm{e}^{r_0(x,y)} \prod_{i=1}^{n} p_i(x, y)^{c_i} \tag{25}$$

where $D[r_0]$ is a rational function of (x, y), the p_i are eigenpolynomials of the D operator and the c_i are constants.

This result can be used to assure the applicability of the method presented in [7], where we have dealt with a restricted class of LFOODEs of the type (1).

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